

# Event Triggered Observer–Based Control for Linear Systems with time varying Uncertainties

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## Abstract

In this paper, we investigate the stabilization of a linear plant subject to network constraints, partial state knowledge and time varying bounded parameter uncertainties. An event–triggered version of the Luenberger observer is proposed, and necessary conditions on the uncertainties are given in term of LMI’s to enable output–based stabilization under different triggering strategies. The proposed observer is tested in simulations on a linearized inverted pendulum.

## I. INTRODUCTION

In modern control system it is more and more common to use digital technology, where the control task consists of sampling the outputs of the plant then computing and implementing new actuator signals. The classic way to proceed is to sample in a periodic fashion the output, thus allowing the closed–loop system to be analyzed on the basis of sampled–data systems [1]. In network control system such as vehicle platooning and smart grid, communication between different agent on the network play a big role in the overall stability. Therefore, it can be of use to reduce the communication when sampling is not needed. Recent years have seen the development of a new paradigm where, instead of sampling periodically, i.e. with a time triggered policy, the system is triggered when needed, i.e. using an event triggered policy. A lot of works have been done on this subject, see [2], [18], [19], [16], [6] and references therein, while for an introduction to the topic see [7].

Our main focus in this work is to investigate the impact of the event triggered paradigm on observer–based (i.e. dynamical feedback) control systems. More precisely, the observer–based control problem is considered for linear systems in the presence of model uncertainties. Different kinds of event triggered policies allow practical or asymptotic stability. Furthermore, the result proposed are global in the sense that the stability does not depend on initial condition and initial observation error. As particular case, the proposed results can be obviously applied for full state feedback.

Some studies are available on observer–based controller [3], [4], [12], [17]. In [3] and [4] practical stability is ensured in presence of a disturbance on the plant, while a  $L_\infty$  gain is guaranteed. In [12], the plant and the output are subject to perturbation and practical stability is also guaranteed. In [17], asymptotic stability is obtained in absence of perturbations. In [5], an uncertain plant is used to stabilize a system between two communications, in the case of availability of the full state. To the best of the authors’ knowledge, no result are available when the system under consideration are not linear.

With respect to these previous results, this paper considers the robustness issues of the event triggered observation and control with respect to time varying modeling uncertainties, allowing to state that if the continuous closed loop system is robust, then the uncertain system is stabilizable with the event triggered policy. Moreover, asymptotic stabilization of an uncertain system using an adapted event triggered policy is obtained.

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The structure of the article is the following. In Section II, we present the general framework. In Section III the event triggered policies for practical and asymptotic output-based stabilization are studied. In Section IV, simulations of a linearized inverted pendulum are presented, to illustrate the proposed event triggered observer-based controller. Finally, some concluding remarks are given.

*Notation:*  $\|\cdot\|$  denotes the euclidean norm,  $\|\cdot\|_{\sup}$  denotes the supremum norm, and  $\lambda_{\min}^P$  ( $\lambda_{\max}^P$ ) is the smallest (biggest) real part of the eigenvalues of the matrix  $P$ .

## II. PROBLEM FORMULATION AND DEFINITIONS

Consider a linear system subject to time varying uncertainties

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t)\end{aligned}\tag{1}$$

where  $x \in \mathbb{R}^n$  is the state,  $u \in \mathbb{R}^m$  is the control,  $y \in \mathbb{R}^p$  is the output,  $\forall t$   $A(t) \in \mathbb{R}^{n \times n}$ ,  $B(t) \in \mathbb{R}^{m \times n}$ ,  $C(t) \in \mathbb{R}^{p \times n}$ , and where  $(A(t), B(t), C(t))$  are subject to parametric uncertainties. System (1) can be rewritten in the following form

$$\begin{aligned}\dot{x}(t) &= (A_0 + \Delta_1(t))x(t) + (B_0 + \Delta_2(t))u(t) \\ y(t) &= (C_0 + \Delta_3(t))x(t)\end{aligned}\tag{2}$$

where  $A_0, B_0, C_0$  represent the nominal matrices, and  $\Delta(t) = (\Delta_1(t), \Delta_2(t), \Delta_3(t))$  represent the time varying uncertainty. The pair  $(A_0, B_0)$  is stabilizable, and the pair  $(A_0, C_0)$  is detectable. In the following, the indication of the time instant  $t$  is dropped if there are no ambiguities.

The control scheme is shown in Fig. 1. Due to the *communication constraints*, there is no continuous communication either between the sensor and the controller, or between the controller and the plant. The value  $y(t_k) = Cx(t_k)$  is available for the controller to implement the control, and the value  $u(t_k)$  is applied to the system, through a classic zero order holder  $H_0$ . It is worth noting that this means that the output  $y$  and the input  $u$  are sampled synchronously, as assumed in this paper for the sake of simplicity, even though generalizations can be done in the case of asynchronous sampling.

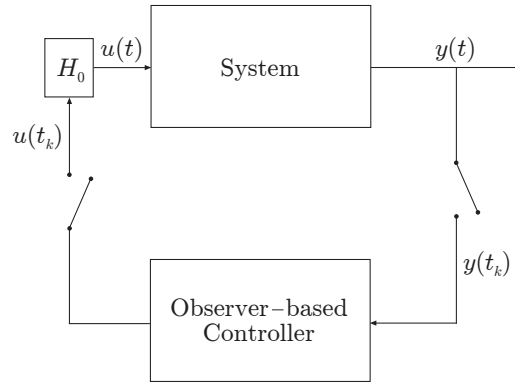


Fig. 1. Control scheme with sampled output and zero order holder

Let us consider first a simple case in which the state  $x$  is available for measuring, and let us assume that there exists a state-feedback  $u = Kx$ ,  $K \in \mathbb{R}^{m \times n}$ , rendering system (1) asymptotically stable at the origin, i.e.  $A + BK$  Hurwitz. When the controller is implemented making use of the sampled value  $x(t_k)$  of the state, one considers the last communication time  $t_k$  between sensors and controller, and the control value  $u(t_k) = Kx(t_k)$ .

### A. Event Triggered Policies

We will first recall some known fact and terminologies about event triggered systems.

Using a classic periodic sampling, the next sampling time is  $t_{k+1} = t_k + \delta$ , where  $\delta > 0$ , so that  $t_{k+1} - t_k = \delta > 0$  or, that is the same

$$t_{k+1} = \min_t \{t \mid t > t_k + \delta\}.$$

The event triggered paradigm [2] replaces this condition with a condition on the state values  $x(t), x(t_k)$ . A simple condition of this kind is, for instance, the epsilon crossing policy, which is of the form

$$t_{k+1} = \min_t \{t > t_k \mid \|x(t) - x(t_k)\| > \varepsilon\}$$

viz.  $x(t)$  is sampled when  $\|x(t) - x(t_k)\|$  is greater than a certain threshold value  $\varepsilon \in \mathbb{R}$ . When this condition is verified, an event is triggered, which determines the sampling time  $t_{k+1}$ . The difference  $\delta_k = t_{k+1} - t_k$  is usually called the inter-event time. To avoid Zeno behaviors [9], it is important that the chosen sampling policy ensures that  $\delta_k > 0$  for all  $k \in \mathbb{N}$ , possibly under additional conditions.

Further strategies can also be used to determine the next sampling time. For instance, the state dependent triggering condition

$$t_{k+1} = \min_t \{t \geq t_k \mid \|x(t) - x(t_k)\| > \sigma \|x\| + \varepsilon\}$$

with  $\varepsilon, \sigma \in \mathbb{R}^+$ , or a mixed triggered policy

$$t_{k+1} = \min_t \{t \geq t_k + \delta_{\min}, \mid \|x(t) - x(t_k)\| > \varepsilon\}$$

with  $\varepsilon, \delta_{\min} \in \mathbb{R}^+$ . Furthermore, (1) can be stabilized asymptotically with the state triggering condition

$$t_{k+1} = \min_t \{t > t_k, \mid \|x(t) - x(t_k)\| > \sigma \|x(t)\|\}$$

if and only if  $(A, B)$  is stabilizable [18].

### B. Problem Formulation

Let us consider the linear system (1). When the state  $x$  is not measurable, under the condition of detectability of the pair  $(A_0, C_0)$  it is possible to build a state observer [15] of the following form. In view of an implementation with a triggered policy, generically the observer has the structure

$$\dot{\hat{x}} = A_0 \hat{x} + B_0 u + GCx(t_k) - GC_0 \hat{x}. \quad (3)$$

A feedback controller based on  $\hat{x}$  will be used in the following to stabilize the system (1). The input applied to the system, after sampling, is

$$u(t) = K \hat{x}(t_k), \quad \forall t \in [t_k, t_{k+1}) \quad (4)$$

so obtaining the controlled dynamics

$$\dot{x} = (A_0 + \Delta_1(t))x + (B_0 + \Delta_2(t))K \hat{x}(t_k). \quad (5)$$

Moreover, considering the sampled value of the output, one gets the following closed-loop system

$$\begin{aligned} \dot{x} &= (A_0 + \Delta_1(t))x + (B_0 + \Delta_2(t))K \hat{x}(t_k) \\ \dot{\hat{x}} &= A_0 \hat{x} + B_0 K \hat{x}(t_k) + Gy(t_k) - GC_0 \hat{x}. \end{aligned} \quad (6)$$

*Definition 2.1:* The origin of a system

$$\dot{\xi} = f(\xi, u, d), \quad \xi \in \mathbb{R}^{\bar{n}}, \quad u \in \mathbb{R}^{\bar{p}} \quad d \in \mathbb{R}^{\bar{m}} \quad (7)$$

is globally ultimately bounded if there is a time  $T_{\xi(0),\varepsilon}$  such that

$$\|\xi(t)\| \leq \varepsilon \quad \forall t > T_{\xi(0),\varepsilon}, \quad \forall \xi(0) \quad (8)$$

for some  $\varepsilon > 0$ . The origin of (7) is *practically stable* if there is a time  $T_{\xi(0),\varepsilon}$  such that (8) holds for any  $\varepsilon > 0$ .  $\diamond$

**Definition 2.2 (Global Minimum Inter–Event Time):** The closed loop system (5) with a triggered policy determining  $t_{k+1}$  is said to have global minimum inter–event time if there exists a time  $\delta_{\min} > 0$  such that  $t_{k+1} - t_k > \delta_{\min} \quad \forall k \in \mathbb{N}$ .  $\diamond$

In the following sections we will address the problems of *output–based practical and asymptotic stabilization*, namely the problem of practical/asymptotic stabilization of the origin of (2) by means of an observed–based controller (3), (4), when appropriate event triggered policies are adopted.

### III. EVENT TRIGGERED POLICIES FOR OUTPUT–BASED STABILIZATION

When the state  $x$  of (1) is not available, the triggered policies reviewed in Section II–A cannot be implemented directly. In the following, we introduce the triggered policy that will be used when the system state is not available. The following definition of mixed triggering for practical stabilization of an observer–based controllers will be used.

**Definition 3.1 (Mixed Triggering for Practical Stability):** The next sampling time for the control is

$$t_{c,k+1} = \max\{t_k + \tau_{\min}, \min_t\{t > t_k \mid \|\hat{x} - \hat{x}(t_k)\| \geq \varepsilon_1\}\} \quad (9)$$

and for the observation is

$$t_{o,k+1} = \max\{t_k + \tau_{\min}, \min_t\{t > t_k \mid \|y - y(t_k)\| \geq \varepsilon_2\}\} \quad (10)$$

where  $\varepsilon_1, \varepsilon_2, \tau_{\min} > 0$ , and

$$t_{k+1} = \min\{t_{c,k+1}, t_{o,k+1}\} \quad (11)$$

is the next sampling time for the closed–loop system, with  $t_0 = 0$  and  $k \in \mathbb{N}$ .  $\diamond$

The proposed triggering condition prevents pathological sampling to appear. In particular, the condition  $t \geq t_k + \tau_{\min}$  in (9), (10) will ensure the absence of Zeno behavior. The triggered policy (11) has obviously a global minimum inter–event time. In the following it will be shown that implementing (11) will ensure the practical solution of the observed–based control problem. However, this condition cannot in general lead to asymptotic stability. For, the following definition of mixed triggering for asymptotic stabilization of an observer–based controller has to be considered.

**Definition 3.2 (Mixed Triggering for Asymptotic Stability):** The next sampling time for the control is

$$t_{c,k+1} = \max\{t_k + \tau_{\min}, \min_t\{t > t_k \mid \|\hat{x} - \hat{x}(t_k)\| \geq \sigma_1 \|\hat{x}\|\}\} \quad (12)$$

and for the observation is

$$t_{o,k+1} = \max\{t_k + \tau_{\min}, \min_t\{t > t_k \mid \|y - y(t_k)\| \geq \sigma_2 \|y\|\}\} \quad (13)$$

where  $\sigma_1, \sigma_2, \tau_{\min} > 0$ , and

$$t_{k+1} = \min\{t_{c,k+1}, t_{o,k+1}\} \quad (14)$$

is the next sampling time for the closed–loop system, with  $t_0 = 0$  and  $k \in \mathbb{N}$ .  $\diamond$

**Remark 3.3:** The existence of  $\tau_{\min} > 0$  can be seen both as a physical constraint and as a requirement in the triggering condition.  $\diamond$

**Remark 3.4:** The synchronization of the triggering condition on the observer and on the output are, *a priori*, not required in order to demonstrate the proposed result. However they allow a much simpler analysis.  $\diamond$

### A. Assumption on the Parameter Uncertainties $\Delta(t)$

Introducing the estimation error  $e = x - \hat{x}$ , from (6) we obtain the following

$$\begin{aligned}\dot{x} &= (A_0 + \Delta_1(t))x + BK(x(t_k) - e(t_k)) \\ \dot{e} &= (A_0 - GC_0)e - GC_0(x(t_k) - x) + \Delta_1(t)x \\ &\quad + \Delta_2(t)K(x(t_k) - e(t_k)) - G\Delta_3(t)x(t_k).\end{aligned}\tag{15}$$

Hence, introducing  $\bar{x} = x - x(t_k)$ ,  $\bar{e} = e - e(t_k)$ , we can rewrite the system in the following form

$$\begin{pmatrix} \dot{\bar{x}} \\ \dot{\bar{e}} \end{pmatrix} = H(\Delta) \begin{pmatrix} \bar{x} \\ \bar{e} \end{pmatrix} + w_p\tag{16}$$

where

$$\begin{aligned}H(\Delta(t)) &= \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \\ H_{11} &= A_0 + \Delta_1(t) + (B_0 + \Delta_2(t))K \\ H_{12} &= -(B_0 + \Delta_2(t))K \\ H_{21} &= \Delta_1(t) + \Delta_2(t)K - G\Delta_3(t) \\ H_{22} &= A_0 - GC_0 - \Delta_2(t)K \\ w_p &= \begin{pmatrix} -(B_0 + \Delta_2(t))K(\bar{x} - \bar{e}) \\ -\Delta_2(t)K(\bar{x} - \bar{e}) + G(C_0 + \Delta_3(t))\bar{x} \end{pmatrix}.\end{aligned}$$

We denote  $h_{ij}(t)$  the element of  $H(\Delta(t))$ .

Since  $\Delta(t)$  represents the parametric uncertainty, it is natural to assume that it is small in a certain sense.  $K, G$  are chosen to render  $H(0)$  We make the following hypothesis on  $H(\Delta(t))$ .

*Assumption 3.5:*  $H(\Delta(t)) \in \mathcal{D}$ , a convex compact set given by  $h_{ij}^{\min} \leq h_{ij}(t) \leq h_{ij}^{\max}$ ,  $\forall i, j, \forall t$ , and there exist  $P = P^T > 0$ ,  $\gamma > 0$  such that

$$H(\Delta(t))^T P + PH(\Delta(t)) + \gamma I_d < 0. \quad \diamond$$

*Remark 3.6:* Given  $\mathcal{D}$ , the hypothesis of a common quadratic Lyapunov function can be checked by a finite set of Linear Matrix Inequality. Using the convexity principle [14], checking the infinite LMIs in Assumption 3.5 reduces to check the LMI on the extremal point of  $\mathcal{D}$ .

*Remark 3.7:* By a continuity argument around  $\Delta(t) = 0$ , If  $H(0)$  is detectable there exist a  $\mathcal{D}$  is small enough (i.e. for the chosen  $K, G$ ) then Assumption 3.5 is verified.  $\diamond$

*Remark 3.8:* Assumption 3.5 implies a bound on  $\Delta_1, \Delta_2 K, G\Delta_3$ . This assumption is less restrictive than a bound on  $\Delta$ .

According to Remark 3.7, one can get sufficient conditions for the stability of the perturbed system from the properties of the nominal system and of the perturbation size.

### B. Practical Stability

In this section we show that if assumption 3.5, then we can stabilize the system in an arbitrary small neighbourhood of the origin. We show that the triggering paradigm can be applied to a system subject to the considered time varying parametric uncertainties.

*Theorem 3.9:* Under Assumption 3.5, it is possible to choose  $\varepsilon_1, \varepsilon_2, \tau_{\min} > 0$  so that the observed-based controller (3), (4), with the triggering condition (11), solves the output-based stabilization problem for system (1).  $\diamond$

*Proof:* We show that, for any  $\varepsilon > 0$ , it is possible to choose  $\varepsilon_1, \varepsilon_2 > 0$  in (9), (10), respectively, such that the triggering condition (11) implies the asymptotic stability toward the ball of radius  $\varepsilon$ . The triggering condition ensures that  $t_{k+1} \geq t_k + \tau_{\min}$ .

From assumption 3.5 it is possible to determine  $K, G$  such that, system (16) is an asymptotically stable system forced by a non vanishing perturbation  $w_p$  due to the difference between  $y$  and its sampled value  $y(t_k)$ , and between  $\hat{x}$  and its sampled value  $\hat{x}(t_k)$ .

*First case*  $t_{k+1} > t_k + \tau_{\min}$ . Since condition (11) ensures  $\|\bar{x} - \bar{e}\| < \varepsilon_1$  and  $\|(C_0 + \Delta_3(t))\bar{x}\| < \varepsilon_2$ , therefore

$$\|w_p\| \leq \|(B_0 + 2\Delta_2(t))K\|\varepsilon_1 + \|G\|\varepsilon_2.$$

Since  $w_p$  is non-vanishing, the trajectories of (16) converge in finite time, depending on the initial condition  $(x(0), e(0))$ , to a ball of the origin bounded by

$$b = \sqrt{\frac{\lambda_{\max}^P}{\lambda_{\min}^P} \frac{\|P\|}{\vartheta\gamma}} \left[ (\|B_0\| + 2\Gamma)\|K\|\varepsilon_1 + \|G\|\varepsilon_2 \right]$$

(see[11]) where  $P = P^T > 0, \gamma$  are defined in assumption 3.5. Here  $\Gamma$  is an upper-bound on  $\Delta_2$ . It is clear that it is always possible to choose  $\varepsilon_1, \varepsilon_2$  such that  $b \leq \varepsilon$ .

*Second case*  $t_{k+1} = t_k + \tau_{\min}$ , let us consider the system

$$\begin{pmatrix} \dot{x} \\ \dot{e} \\ \dot{\bar{x}} \\ \dot{\bar{e}} \end{pmatrix} = \bar{A}(t) \begin{pmatrix} x \\ e \\ \bar{x} \\ \bar{e} \end{pmatrix} = \begin{pmatrix} H(\Delta)(t) & \bar{A}_{12} \\ H(\Delta)(t) & \bar{A}_{12} \end{pmatrix} \begin{pmatrix} X \\ \bar{X} \end{pmatrix} \quad (17)$$

with  $X = (x^T, e^T)^T, \bar{X} = (\bar{x}^T, \bar{e}^T)^T, \bar{x} = x - x(t_k), \bar{e} = e - e(t_k)$ , and

$$\bar{A}_{12} = \begin{pmatrix} -(B_0 + \Delta_2)K & +(B_0 + \Delta_2)K \\ -G(C_0 + \Delta_3) - \Delta_2K & \Delta_2K \end{pmatrix}.$$

Setting  $z = (X^T, \bar{X}^T)^T$  At the sampling instants we have,  $z(t_k) = (X^T(t_k), 0^T)^T$ . Let  $\mathcal{T}$  be the projection of  $z$  on  $\bar{X}$  so that  $\mathcal{T}z = \bar{X}$ , introducing the  $\Phi(t, t_0)$  resolvent of (17) we denote  $\|\Phi(t, t_0)\|_{\sup} = \max_{\|x\|_{\sup}=1} \|\Phi(t, t_0)x\|_{\sup}$

we can write

$$z(t) = \Phi(t, t_k)z(t_k) = \Phi(t, t_k) \begin{pmatrix} X(t_k) \\ 0 \end{pmatrix}, \quad t \in [t_k, t_{k+1})$$

Between two sampling instants we have

$$\begin{aligned} \bar{X}(t_{k+1}^-) &= \bar{X}(t_{k+1}^-) - \bar{X}(t_k^+) \\ &= \mathcal{T}(\Phi(t, t_k) - I_d) \begin{pmatrix} X(t_k) \\ 0 \end{pmatrix}. \end{aligned}$$

Furthermore  $\Phi(t, t) = I_d$  and from assumption 3.5 where  $\mathcal{D}$  is a compact set, between two sampling

$$e^{-M_{\Phi}(t_2-t_1)} \leq \|\Phi(t_2, t_1)\|_{\sup} \leq e^{M_{\Phi}(t_2-t_1)} \quad (18)$$

with  $M_{\Phi}$  a positive constant. Hence,

$$\|\bar{X}(t)\|_{\sup} \leq \|\mathcal{T}\| \|\Phi(t, t_k) - I_d\|_{\sup} \|X(t_k)\|_{\sup}, \quad t \in [t_k, t_{k+1}).$$

then (18) imply

$$\|\bar{X}(t)\|_{\sup} \leq \|\mathcal{T}\|_{\sup} (e^{M_{\Phi}(t-t_k)} - 1) \|X(t_k)\|_{\sup}$$

with  $\|\mathcal{T}\|_{\sup} = 1$  since it is a projection. Furthermore from (18)

$$\|X(t_k)\|_{\sup} \leq \|X(t)\|_{\sup} e^{M_{\Phi}(t-t_k)}.$$

Therefore, for all  $t \in [t_k, t_{k+1})$

$$\|\bar{X}(t)\|_{\sup} \leq (e^{M_{\Phi}(t-t_k)} - 1)e^{M_{\Phi}(t-t_k)}\|X(t)\|_{\sup}$$

and hence

$$\|\bar{X}(t)\|_{\sup} \leq (e^{M_{\Phi}\tau_{\min}} - 1)e^{M_{\Phi}(\tau_{\min})}\|X(t)\|_{\sup}, \quad t \in [t_k, t_{k+1}).$$

Considering

$$\rho(\tau_{\min}) = (e^{M_{\Phi}\tau_{\min}} - 1)e^{M_{\Phi}\tau_{\min}}$$

for any  $\epsilon > 0$  there exists  $\tau_{\min}$  small enough so that  $\rho(\tau_{\min}) \leq \epsilon$ , since between two sampling instants  $\|\bar{X}\|_{\sup} \leq \rho\|X\|_{\sup}$  and (3.5) is verified. From norm equivalence we have  $\rho' = C\rho$  such that  $\|\bar{X}\| \leq \rho\|X\|$ . In conclusion, considering the Lyapunov candidate  $V = X^T P X$ , with

$$\dot{V} \leq -\gamma\|X\|^2 + \rho'(\tau_{\min})\|P\bar{A}_{12}\|\|X\|^2$$

One can always choose  $\tau_{\min}$  such that

$$-\gamma + \rho'(\tau_{\min})\|P\bar{A}_{12}\| \leq -\sigma$$

for any  $\sigma < \gamma$  implying that  $(x^T, e^T)^T$  converges to zero. i.e.  $\bar{X}$  can be seen as a vanishing perturbation affecting  $X$ .

So as long as  $t_{k+1} - t_k = \tau_{\min}$ ,  $(x^T, e^T)^T$  converges to zero asymptotically (exponentially), and when  $t_{k+1} - t_k > \tau_{\min}$  the system goes toward a ball of radius  $\epsilon$ . Therefore the proposed triggering condition leads to practical stability.  $\blacksquare$

### C. Asymptotic Stability

Theorem 3.9 ensures only practical stability, despite the fact that asymptotic stability of the continuous closed loop system is assumed. Nevertheless, asymptotic stability can be recovered changing the triggering condition (11) with (14), as stated by the following.

*Theorem 3.10:* Under assumption 3.5, it is possible to choose  $\sigma_1, \sigma_2, \tau_{\min} > 0$  so that the observed-based controller (3), (4), with the triggering condition (14), solves the output-based asymptotic stabilization problem for system (1).  $\diamond$

*Proof:* It was shown in the proof of Theorem 3.9 the existence of a  $\tau_{\min}$  ensuring the asymptotic stability of (6) as long as  $t_{k+1} - t_k = \tau_{\min}$ . Now assume  $t_{k+1} - t_k > \tau_{\min}$ , where  $t_k$  is the last triggering time, and let us introduce the extended state  $X = (x^T, e^T)^T$ . One writes

$$\dot{X} = H(\Delta(t))X + \begin{pmatrix} -(B_0 + \Delta_2(t))K(\bar{x} - \bar{e}) \\ -\Delta_2(t)K(\bar{x} - \bar{e}) - G(C_0 + \Delta_3(t))\bar{x} \end{pmatrix}$$

where the definitions of  $\bar{x}, \bar{e}$  are as in the proof of Theorem 3.9. There exists  $P = P^T > 0$  such that  $= H(\Delta(t))^T P + PH(\Delta(t)) + \gamma Id < 0, \forall t$ . Considering  $V = X^T P X$

$$\begin{aligned} \dot{V} &= X^T (H(\Delta(t))^T P + PH(\Delta(t))) X \\ &\quad + 2X^T P \begin{pmatrix} -(B_0 + \Delta_2(t))K(\bar{x} - \bar{e}) \\ -\Delta_2(t)K(\bar{x} - \bar{e}) - G(C_0 + \Delta_3(t))\bar{x} \end{pmatrix} \\ &\leq -\gamma\|X\|^2 \\ &\quad + 2\|X\|\|P\| \left\| \begin{pmatrix} -(B + \Delta_2(t))K(\bar{x} - \bar{e}) \\ -\Delta_2(t)K(\bar{x} - \bar{e}) - G(C_0 + \Delta_3(t))\bar{x} \end{pmatrix} \right\| \end{aligned}$$

The triggering condition (14) implies that

$$\|\bar{x} - \bar{e}\| \leq \sigma_1\|x - e\|, \quad \|C\bar{x}\| \leq \sigma_2\|(C_0 + \Delta_3(t))x\|.$$

Since

$$\begin{aligned} & \left\| \begin{pmatrix} -(B + \Delta_2(t))K(\bar{x} - \bar{e}) \\ \Delta_2(t)K(\bar{x} - \bar{e}) - G(C_0 + \Delta_3(t)(\bar{x})) \end{pmatrix} \right\| \leq 2 \max\{a_1, a_2\} \\ a_1 &= \sigma_1 \|(B + \Delta_2(t))K\| \|x - e\| \\ a_2 &= \sigma_2 \|G\| \|e\| + \Delta_2(t)K \|x - e\| \end{aligned}$$

and  $\|e\| \leq \|X\|$ ,  $\|x - e\| \leq 2\|X\|$ , one finally works out

$$\dot{V} \leq \|X\|^2 \left( -\gamma + 4\|P\|s(\sigma_1, \sigma_2) \right)$$

with

$$s(\sigma_1, \sigma_2) = \max \left\{ \|(B + \Delta_2(t))K\| \sigma_1, 2\|G\| \sigma_2 + \Delta_2(t)K \sigma_1 \right\}.$$

It is always possible to choose  $\sigma_1, \sigma_2$  such that

$$-\gamma + 4\|P\|s(\sigma_1, \sigma_2) < 0.$$

This guarantees the (exponential) asymptotic stability of the extended system. ■

*Remark 3.11:* The proof of Theorem 3.10 shows that the triggered policy (14), while enforcing asymptotic stability, can ensure slower convergence rates. ◇

#### IV. NUMERICAL SIMULATION RESULTS

Let us consider system (1), with

$$\begin{aligned} A &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{(I + ml^2)b}{p} & \frac{m^2 gl^2}{p} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{mlb}{p} & \frac{mgl(M + m)}{p} & 0 \end{pmatrix} \\ B &= \begin{pmatrix} 0 & \frac{I + ml^2}{p} & 0 & \frac{ml}{p} \end{pmatrix}^T, \quad C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{aligned}$$

representing by the linearization of the inverted pendulum on a cart, with  $x_1$  the cart position,  $x_2$  its velocity,  $x_3$  the pendulum angle, and  $x_4$  its angular velocity. Clearly,  $\Delta_3 = 0$ . The nominal parameter values are

$$\begin{aligned} l_0 &= 0.3 \text{ m}, & m_0 &= 0.5 \text{ Kg}, & M_0 &= 0.5 \text{ Kg} \\ I_0 &= 0.006 \text{ Kg m}^2, & b_0 &= 0.1 \text{ Kg/s}, & g_0 &= 9.8 \text{ m/s}^2 \end{aligned}$$

and  $p_0 = I_0(M_0 + m_0) + M_0 m_0 l_0^2$ , while the real parameters are

$$\begin{aligned} l &\in [0.27, 0.33] \text{ m} & m &\in [0.45, 0.55] \text{ Kg} \\ M &\in [0.45, 0.55] \text{ Kg} & I &\in [0.0056, 0.0064] \text{ Kg m}^2 \\ b &= b_0 \text{ Kg/s}, & g &= 9.8 \text{ m/s}^2 \end{aligned}$$

and  $p = I(M + m) + Mml^2$ . The matrices  $K$  and  $L$  are chosen such that the nominal system has its biggest eigenvalue equal to  $-2$ .

The simulations results are shown in Fig. 2. The initial conditions are  $x(0) = (0.1, 0, 0.2, 0)$ , and  $\hat{x}(0) = 0$ . Fig. 2 refers to a simulation in which the triggering condition (11) is used, with  $\tau_{\min} = 10^{-3}$  s,  $\varepsilon_1 = \varepsilon_2 = 10^{-3}$ . The practical stability can be observed in Fig. 2.a, and the inter-event time is shown in Fig. 2.b. While during the transient phase the inter-event times are  $\tau_{\min}$ , the successive inter-event times are determined by  $\varepsilon_1, \varepsilon_2$ . Fig. 3 refers with the triggering condition (14), with  $\tau_{\min} = 10^{-3}$  s,  $\sigma_1 = \sigma_2 = 0.05$ . This time, one observes asymptotic stability (Fig. 3.a), with a shorter average inter-event time (Fig. 3.b). The imposed minimum inter-event time does not turn to be useful in this instance of the problem since the output norm and the observer norm never cross 0



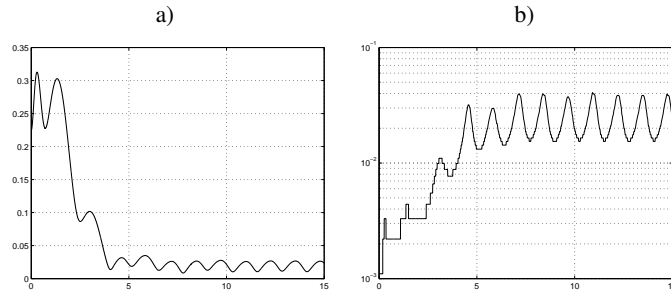


Fig. 2. Triggering condition (11) a) Euclidean norm of the state vs time; b) Inter-event time  $t_{k+1} - t_k$  vs time.

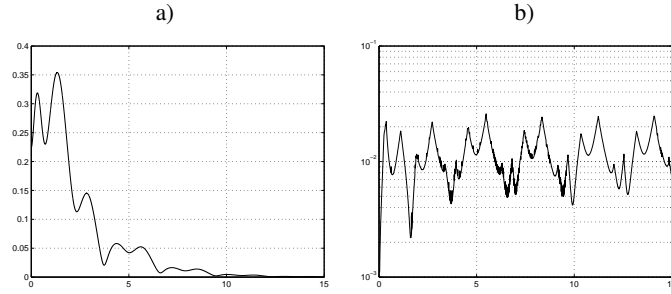


Fig. 3. Triggering condition (14) a) Euclidean norm of the state vs time; b) Inter-event time  $t_{k+1} - t_k$  vs time.

## V. CONCLUSIONS

We have shown that for linear systems with time varying parametric uncertainties, and in the case of partial state knowledge, the event triggered paradigm can successfully be applied for stabilization via observer-based controllers. Practical and asymptotic stability are demonstrated when considering model uncertainties. The proposed results recover full state feedback as a special case. We have shown the applicability of the proposed approach considering the linearized dynamics of the inverted pendulum on a cart. Further studies should include practical method of calculating an optimal choice of the triggering parameters, robustness with respect to disturbance and measurement noise.

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